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Diagonal Harmonious State Representations

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For simple unweighted shift operators a family of complex eigenvalue eigenstates of the shift down operators, called the *harmonious states*, is constructed. Every density matrix is realized as a weighted sum of projections to the harmonious states; and the weight distributions serve as quasiprobability densities for normal ordered operators.

1. INTRODUCTION

Coherent states for canonical systems (Schrödinger, 1933) have been found to be useful in quantum optics and in discussions of quantum mechanics in general (Klauder and Sudarshan, 1968). They may be defined in a number of more or less equivalent ways; as states obtaining the minimum for the Heisenberg uncertainty relations, as the family obtained by translating the harmonic oscillator vacuum in coordinate and momentum variables, as the normalized eigenstates of the annihilation operator for complex eigenvalues, and so on. If we have the operators q, p satisfying the Weyl commutation relations (Weyl, 1931)

$$\exp(i\lambda p)\exp(i\mu q)=\exp(i\lambda\mu)\exp(i\mu q)\exp(i\lambda p) \tag{1}$$

or the canonical commutation relations with

$$a=\frac{q+ip}{\sqrt{2}}; \qquad qp-pq=i\mathbb{1}; \qquad aa^{\dagger}-a^{\dagger}a=1 \tag{2}$$

then there exists a vacuum state $|0\rangle$ such that $a\,|0\rangle=0$.

The Fock states

$$|n\rangle=\frac{(a^{\dagger})^n}{(n!)^{1/2}}\,|0\rangle=\left(\frac{q-ip}{\sqrt{2}}\right)^n\frac{1}{(n!)^{1/2}}\,|0\rangle \tag{3}$$

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form an orthonormal complete basis:

$$\langle m | n \rangle = \delta_{mn}; \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \tag{4}$$

For any complex number z the coherent states are defined by

$$\begin{aligned} |z\rangle &= e^{i(za^\dagger - z^* a)} |0\rangle = T(z, z^*) |0\rangle \\ &= e^{-|z|^2/2} e^{za^\dagger} |0\rangle = e^{-z^* z/2} \sum_0^{\infty} \frac{z^n}{(n!)^{1/2}} |n\rangle \end{aligned} \tag{5}$$

they are normalized:

$$\langle z | z \rangle = 1 \tag{6}$$

since the generic displacement operator $T(z, z^*)$ is unitary. One can also verify the statement by direct calculations. The displacement operators $T(z, z^*)$ furnish a unitary projective realization of the displacements in q and p and hence in a and $a^\dagger$:

$$\begin{aligned} &T(z_1, z_1^*)\, T(z_2, z_2^*) = \exp\left(\frac{z_1 z_2 - z_1^* z_2}{2}\right) T(z_1 + z_2, z_1^* + z_2^*) \\ &T(z, z^*)\, a T^\dagger(z, z^*) = a - z \\ &T(z, z^*)\, a^\dagger T^\dagger(z, z^*) = a^\dagger - z \end{aligned} \tag{7}$$

2. THE DIAGONAL COHERENT STATE REPRESENTATION

The states $|z\rangle$ are uncountably infinite and hence are not all linearly independent. If $F(z)$ is any entire function of z, then

$$\int d^2 z\, z F(z)\, |z\rangle = 0 \tag{8}$$

They are not mutually orthogonal since

$$\langle z_1 | z_2 \rangle = \exp\{-\tfrac{1}{2}(z_1^* z_1 - 2 z_1^* z_2 + z_1^*)\} \tag{9}$$

But these states are complete; and we have

$$\frac{1}{\pi} \int d^2 z\, |z\rangle\langle z| = 1 \tag{10}$$

and therefore

$$|\psi\rangle = \frac{1}{\pi} \int d^2 z\, \langle z | \psi \rangle\, |z\rangle \tag{11}$$

The projection operator $|x+iy\rangle\langle x+iy|$ is analytic in the variables x, y and a distribution-weighted linear combination of them can give a diagonal representation of any bounded operator; in particular, for the density matrix the diagonal representation (Sudarshan, 1963) is

$$\rho = \frac{1}{\pi}\int d^2z\, R(z)\, |z\rangle\langle z| \tag{12}$$

with a real weight $R(z)$. In general $R(z)$ is not a smooth function or even a tempered distribution. The Fourier transform of $R(z)$ is given by (Mehta and Sudarshan, 1965)

$$\begin{aligned}\tilde{R}(\zeta) &\equiv \frac{1}{2\pi}\int \exp(\zeta z^* - \zeta^* z)\, R(z)\, d^2z \\ &= \exp(\zeta^*\zeta)\, \frac{1}{2\pi}\int \exp(\zeta z^* - \zeta^* z)\, \langle z|\, \rho\, |z\rangle\, d^2z \end{aligned} \tag{13}$$

from which $R(z)$ can be determined as a suitable distribution.

The diagonal representation can be expanded in the Fock basis:

$$\begin{aligned}&\frac{1}{\pi}\int R(z, z^*)\, |z\rangle\langle z|\, d^2z \\ &\quad = \sum_{m,n} \int e^{z^* z}\, \frac{R(z, z^*) z^m z^{*n}}{\pi(m!n!)^{1/2}}\, |m\rangle\langle n|\, d^2z \\ &\quad = \sum_{m,n} \rho_{mn}\, |m\rangle\langle n| \end{aligned} \tag{14}$$

so that

$$\rho_{mn} = \frac{1}{\pi}\int d^2z\, e^{-z^* z}\, \frac{R(z, z^*) z^m z^{*n}}{(m!n!)^{1/2}} \tag{15}$$

By expanding $R(z, z^*)$ in terms of harmonic series in $(z/z^*)^{1/2} = e^{i\theta}$, we may put

$$R(z, z^*) = \sum_{\nu}^{\infty} R_\nu(\gamma) e^{i\nu\theta}, \qquad \gamma = (z^* z)^{1/2} \tag{16}$$

with a corresponding decomposition of ρ. Since

$$\rho_{mn} = \langle m|\, \rho\, |n\rangle \tag{17}$$

we get, using

$$|m\rangle\langle n| = \frac{1}{2\pi} \iint e^{i(n-m)\theta} \frac{\delta^{(m+n+1)}(r)(-1)^{(m+n+1)}}{(m+n+1)!} |re^{i\theta}\rangle\langle re^{i\theta}| \, r \, dr \, d\theta \tag{18}$$

the density matrix in the form

$$\rho = \frac{1}{2\pi} \sum_{m,n} \iint \langle m| \rho |n\rangle e^{i(n-m)\theta} \left(-\frac{\partial}{\partial r}\right)^{m+n+1} \delta(r) \, |re^{i\theta}\rangle\langle re^{i\theta}| \, r \, dr \, d\theta \tag{19}$$

We recognize that the distributions $\{(-\partial/\partial r)^N \delta(r)\}$ are dual to $\{r^N\}$ (Sudarshan, 1963):

$$\left((-)^N \frac{\partial^{(N)}}{\partial r^N} \delta(r), r^{N'}\right) = \delta_{N,N'} \tag{20}$$

Thus we get a set of linear functionals dual to the monomials. The diagonal coherent state weight is a linear sum of such duals with harmonic factors, since with the measure $(1/\pi)\, d^2z = d\mu(z)$ the monomials $z^m z^{*n}$ have the duals

$$e^{i(n-m)\theta} \left(-\frac{\partial}{\partial r}\right)^{m+n+1} \delta(r)$$

3. UNWEIGHTED SHIFT OPERATORS

The creation operator $a^\dagger$ is an example of a weighted shift since

$$a^\dagger |n\rangle = \omega_n |n+1\rangle, \qquad \omega_n = (n+1)^{1/2} \tag{21}$$

and may be viewed as the product of a Hermitian operator $(a^\dagger a)^{1/2}$ and an isometry

$$a^\dagger = (a^\dagger a)^{1/2} b^\dagger; \qquad b^\dagger |n\rangle = (n+1) \tag{22}$$

The quantity $(a^\dagger a)^{1/2}$ is defined as the positive square root of $a^\dagger a$ which here takes on the values 1, 2, 3, ... While $b^\dagger$ is isometric, we cannot define an inverse, since the operator

$$b = (b^\dagger)^\dagger = a(a^\dagger a)^{-1/2} \tag{23}$$

is not defined on the state $|0\rangle$ unless we define it to be zero. In that case b is not norm preserving since it annihilates $|0\rangle$. The shift operator $b^\dagger$ in not normal since

$$bb^\dagger - b^\dagger = |0\rangle\langle 0| \tag{24}$$

Consequently we do not expect to diagonalize both the operators b, $b^\dagger$.

The Fock representation of the shift operators

$$b\,|m\rangle = \begin{cases} |n-1\rangle, & n>0 \\ 0, & n=0 \end{cases}$$
$$b^{\dagger}\,|n\rangle = |n+1\rangle, \qquad n \geqslant 0 \tag{25}$$

is in the Jordan block form

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \qquad b^{\dagger} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{26}$$

4. THE HARMONIOUS STATES

There exists a "harmonious state" representation for b which fills the open unit circle in the complex plane. We define

$$|z) = \sum_{0}^{\infty} z^{n}\,|n\rangle; \qquad z^{*}z < 1 \tag{27}$$

These states are normalized for z inside the unit circle. They are not mutually orthogonal

$$(z_1\,|\,z_2) = (1 - z_1^{*} z_2); \qquad |z_1 z_2| < 1 \tag{28}$$

Note that the scalar product is defined even if one of the numbers z_1, z_2 has an arbitrary modulus as long as the product of the moduli is less than unity.

These states are eigenstates of the step-down operator

$$b\,|z) = z\,|z), \qquad (z|\,b^{\dagger} = z^{*}(z| \tag{29}$$

On the other hand, there are no right eigenvectors for $b^{\dagger}$.

To be able to construct a resolution of the identity in terms of the projections $|z)(z|$ we look for a completeness identity; we get

$$\frac{1}{2\pi}\int d\theta\,|re^{i\theta})(re^{i\theta}| = \sum_{n=0}^{\infty} r^{2n}\,|n\rangle\langle n| \tag{30}$$

Hence

$$\lim_{r\to 1}\frac{1}{2\pi}\int d\theta\,|re^{i\theta})(re^{i\theta}| = 1 \tag{31}$$

Thus only the states at the boundary of the unit circle are involved. Since the monomials r^{2n} have their duals

$$(-)^n \frac{(\partial/\partial r^2)^n \, \delta(r^2)}{n!} \tag{32}$$

with respect to the measure $d\mu(r^2, \theta) = (1/2\pi)\, d\theta \, d(r^2)$ we could also write the harmonious state resolution of the identity as

$$\frac{1}{2\pi} \int d\theta \, d(r^2) \sum_{n=0}^{\infty} (-1)^n \frac{(\partial/\partial(r^2))^n}{\delta(r^2)} \, |re^{i\theta})(re^{i\theta}| = 1 \tag{33}$$

Since the operator

$$\sum \frac{(-)^n}{N!} \left(\frac{\partial}{\partial(r^2)}\right)^n \delta(r^2) = e^{+\partial/\partial(r^2)} \delta(r^2) = \delta(r^2 - 1) \tag{34}$$

this yields the identity

$$\begin{aligned} \mathbb{1} &= \frac{1}{\pi} \int d\theta \int r \, dr \, \delta(r^2 - 1) \, |re^{i\theta})(re^{i\theta}| \\ &= \frac{1}{2\pi} \int d\theta \, |e^{i\theta})(e^{i\theta}| \end{aligned} \tag{35}$$

5. THE DIAGONAL HARMONIOUS STATE REPRESENTATION

To obtain a diagonal representation we may proceed as for the Schrödinger coherent states. For any operator ρ go to the Fock basis

$$\rho = \sum_{m,n} |m\rangle\langle n| \, p_{mn} \tag{36}$$

Now

$$\frac{1}{2\pi} \int d\theta \, e^{-i\nu\theta} \, |re^{i\theta})(re^{i\theta}| = \sum_m |m\rangle\langle m + \nu| \, r^{2m+\nu} \tag{37}$$

With the integration measure dr, the dual to r^N is $(-\partial/\partial r)^N \, \delta(r)$. Hence

$$|m\rangle\langle n| = \frac{1}{2\pi} \int d\theta \, e^{i(n-m)\theta} \int r \, dr \left(-\frac{\partial}{\partial r}\right)^{m+n+1} \delta(r) \, |re^{i\theta})(re^{i\theta}| \tag{38}$$

Hence the diagonal harmonious state representation for ρ is

$$\rho = \sum_{m,n} \rho_{m,n} \frac{1}{2\pi} \int d\theta \, e^{i(n-m)\theta} \int r \, dr \left(-\frac{\partial}{\partial r}\right)^{m+n+1} \delta(r) \, |re^{i\theta})(re^{i\theta}| \tag{39}$$

and the weight distribution is

$$R(re^{i\theta})=\frac{1}{2\pi}\int d\theta\int r\,dr\sum_{m,n}e^{i(n-m)\theta}\langle m|\,\rho\,|n\rangle\left(-\frac{\partial}{\partial r}\right)^{m+n+1}\delta(r) \quad (40)$$

The diagonal harmonious state representation of the density matrix can serve as a quasiprobability distribution (Sudarshan, 1963). We get

$$\begin{aligned}\langle(b^\dagger)^m\,b^n\rangle&=\mathrm{Tr}((b^\dagger)^m\,b^n\rho)\\&=\frac{1}{2\pi}\int_0^1 r\,dr\int_0^{2\pi}d\theta\,R(re^{i\theta})(re^{-i\theta})^m\,(re^{i\theta})^n\end{aligned} \quad (41)$$

or, more generally, for any normal ordered quantity

$$\langle{:}A(b^\dagger, b){:}\rangle=\frac{1}{2\pi}\int d^2z\,R(z)\,A(z^*, z) \quad (42)$$

Correspondingly, following Husimi (1950) and Kano (1965), for any antinormal ordered operator $B(b, b^\dagger)$

$$\langle B(b, b^\dagger)\rangle=\frac{1}{2\pi}\int d^2z\,(z|\,\rho\,|z)\,B(z, z^*) \quad (43)$$

Some boundedness on the operator is required. Since the natural domain of convergence of a power series is a circle, if the matrix elements increase with m, n, the diagonal weight distribution is not meaningful.

DISCUSSION

We recognize that the projection

$$|z)(z|=|x+iy)(x+iy|$$

is the boundary value of an analytic function in two complex variables ξ, η of an open neighborhood of the origin. Hence the projections in the neighborhood of the origin can be infinitely differentiated. Therefore if $|z)(z|$ are given along any "arc" in the product of these variables, that is, in an open neighborhood of the origin, they form a complete set. It follows that $(z|\,A\,|z)$ for such a neighborhood uniquely defines the operator A. The characteristic sets S, which are such sets for which $(z|\,A\,|z)=0$, $z\in S$, implies $A=0$, are yet to be determined for these harmonious states.

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